DREIBENS MODULO 7

A New Formula for Primality Testing

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IN THIS PAPER, WE DISCUSS STRINGS OF 3'S AND 7'S, HEREBY DUBBED "DREIBENS." AS A FIRST STEP TOWARDS DETERMINING WHETHER THE SET OF PRIME DREIBENS IS INFINITE, WE EXAMINE THE PROPERTIES OF DREIBENS WHEN DIVIDED BY 7. BY DETERMINING THE DIVISIBILITY OF A DREIBEN BY 7, WE CAN RULE OUT SOME COMPOSITE DREIBENS IN THE SEARCH FOR PRIME DREIBENS. WE ARE CONCERNED WITH THE NUMBER OF DREIBENS OF LENGTH N THAT LEAVE A REMAINDER I WHEN DIVIDED BY 7. BY USING NUMBER THEORY, LINEAR ALGEBRA, AND ABSTRACT ALGEBRA, WE ARRIVE AT A FORMULA THAT TELLS US HOW MANY DREIBENS OF LENGTH N ARE DIVISIBLE BY 7. WE ALSO FIND A WAY TO DETERMINE THE NUMBER OF DREIBENS OF LENGTH N THAT LEAVE A REMAINDER I WHEN DIVIDED BY 7. FURTHER INVESTIGATION FROM A COMBINATORIAL PERSPECTIVE PROVIDES ADDITIONAL INSIGHT INTO THE PROPERTIES OF DREIBENS WHEN DIVIDED BY 7. OVERALL, THIS PAPER HELPS CHARACTERIZE DREIBENS, OPENS UP MORE PATHS OF INQUIRY INTO THE NATURE OF DREIBENS, AND RULES OUT SOME COMPOSITE DREIBENS FROM A PRIME DREIBEN SEARCH.

INTRODUCTION TO DREIBENS AND THE SEARCH FOR INFINITELY MANY PRIME DREIBENS

The existence of infinitely many prime numbers is an elementary fact of number theory. However, mathematicians continue to ponder over whether there are infinitely many primes of a certain kind. For instance, it is not yet known whether or not the set of Mersenne primes is finite or infinite. Evidently, questions about primes remain at the forefront of number theory.

Our current research focuses on a type of number that we dubbed *dreibens*. Dreibens are merely strings of 3s and 7s (e.g. 333, 777, 3377373737, etc.), yet these simple strings provide a foundation for a multitude of questions. Our ultimate goal is to prove whether or not infinitely many prime dreibens exist. While that question is beyond the scope of this paper, this paper contributes to that goal by ruling out some composite dreibens. In fact, this paper concerns itself with dreibens divisible by 7; by understanding dreibens modulo 7, we can rule out some dreibens that are certainly *not* prime.

This paper uses a variety of approaches to learn about dreibens modulo 7, including techniques from number theory, abstract algebra, linear algebra, and combinatorics. With time, we hope dreibens to one day become a very well characterized type of number.

LEARNING ABOUT DREIBENS MODULO 7 BY EXPLORING $A_7^i(N)$

We would best approach dreibens by attempting to nd the number of dreibens divisible by some number. For this paper, we interest ourselves in the number of dreibens divisible by 7. Let us define $A_j^i(n)$ as the number 1 of dreibens of length n that leave a remainder i when divided by j. In this paper, we will concern ourselves with $A_7^i(n)$.

How do we find $A_7^i(n+1)$? We can find formulas for $A_7^i(n+1)$ by considering the first n digits of a dreiben of length $n+1$, then seeing what the last digit must be such that the dreiben leaves a remainder i when divided by 7. Let us consider a dreiben D_{n+1} of length $n+1$, which could leave a remainder of 0, 1, 2, 3, 4, 5, or 6 when divided by 7. The rst n digits of D_{n+1} form another dreiben D_n that could leave a remainder of 0, 1, 2, 3, 4, 5, or 6 when divided by 7. Hence, we have seven cases, each with seven sub-cases, which we must consider. We will write the rst three cases below.

Case 1: D_{n+1} leaves a remainder of 0 when divided by 7, i.e. D_{n+1} is counted in $A_7^0(n+1)$.

(a) If D_n is 0 (mod 7), i.e. D_n is counted in $A_7^0(n)$, then the last digit of D_{n+1} must be 7.

(b) If D_n is 1 (mod 7), i.e. D_n is counted in $A_7^1(n)$, then there is no digit that can be appended to D_n such that D_{n+1} is divisible by 7.

(c) If D_n is 2 (mod 7), i.e. D_n is counted in $A_7^2(n)$, then there is no digit that can be appended to D_n such that D_{n+1} is divisible by 7.

(d) If D_n is 3 (mod 7), i.e. D_n is counted in $A_7^3(n)$, there is no digit that can be appended to D_n such that D_{n+1} is divisible by 7.

(e) If D_n is 4 (mod 7), i.e. D_n is counted in $A_7^4(n)$, then there is no digit that can be appended to D_n such that D_{n+1} is divisible by 7.

(f) If D_n is 5 (mod 7), i.e. D_n is counted in $A_7^5(n)$, then there is no digit that can be appended to D_n such that D_{n+1} is divisible by 7.

(g) If D_n is 6 (mod 7), i.e. D_n is counted in $A_7^6(n)$, then the last digit of D_{n+1} must be 3.

Case 2: D_{n+1} leaves a remainder of 1 when divided by 7, i.e. D_{n+1} is counted in $A_7^1(n+1)$.

(a) If D_n is 0 (mod 7), i.e. D_n is counted in $A_7^0(n)$, then there is no digit that can be appended to D_n such that D_{n+1} is 1 (mod 7).

(b) If D_n is 1 (mod 7), i.e. D_n is counted in $A_7^1(n)$, then there is no digit that can be appended to D_n such that D_{n+1} is 1 (mod 7).

(c) If D_n is 2 (mod 7), i.e. D_n is counted in $A_7^2(n)$, then there is no digit that can be appended to D_n such that D_{n+1} is 1 (mod 7).

(d) If D_n is 3 (mod 7), i.e. D_n is counted in $A_7^3(n)$, then there is no digit that can be appended to D_n such that D_{n+1} is 1 (mod 7).

(e) If D_n is 4 (mod 7), i.e. D_n is counted in $A_7^4(n)$, then the last digit of D_{n+1} must be 3.

(f) If D_n is 5 (mod 7), i.e. D_n is counted in $A_7^5(n)$, then the last digit of D_{n+1} must be 7.

(g) If D_n is 6 (mod 7), i.e. D_n is counted in $A_7^6(n)$, then there is no digit that can be appended to D_n such that D_{n+1} is 1 (mod 7).

Case 3: D_{n+1} leaves a remainder of 2 when divided by 7, i.e. D_{n+1} is counted in $A_7^2(n+1)$.

(a) If D_n is 0 (mod 7), i.e. D_n is counted in $A_7^0(n)$, then there is no digit that can be appended to D_n such that D_{n+1} is 2 (mod 7).

(b) If D_n is 1 (mod 7), i.e. D_n is counted in $A_7^1(n)$, then there is no digit that can be appended to D_n such that D_{n+1} is 2 (mod 7).

(c) If D_n is 2 (mod 7), i.e. D_n is counted in $A_7^2(n)$, then the last digit of D_{n+1} must be 3.

(d) If D_n is 3 (mod 7), i.e. D_n is counted in $A_7^3(n)$, then the last digit of D_{n+1} must be 7.

(e) If D_n is 4 (mod 7), i.e. D_n is counted in $A_7^4(n)$, then there is no digit that can be appended to D_n such that D_{n+1} is 2 (mod 7).

(f) If D_n is 5 (mod 7), i.e. D_n is counted in $A_7^5(n)$, then there is no digit that can be appended to D_n such that D_{n+1} is 2 (mod 7).

(g) If D_n is 6 (mod 7), i.e. D_n is counted in $A_7^6(n)$, then there is no digit that can be appended to D_n such that D_{n+1} is 2 (mod 7).

"We can find formulas for $A_7^i(n+1)$ by considering the first n digits of a dreiben of length n + 1, then seeing what the last digit must be such that the dreiben leaves a remainder i when divided by 7."

Notice that in Case 1, D_{n+1} covers all possible dreibens of length $n+1$ divisible by 7. Likewise, Case 2 covers all possible dreibens of length $n+1$ that are 1 (mod 7), Case 3 covers all possible dreibens of length $n+1$ that are 2 (mod 7), and so forth. Similarly, each subcase (a) covers all possible dreibens of length n divisible by 7, each subcase (b) covers all possible dreibens of length n that are 1 (mod 7), and so forth. Hence, each case represents $A_7^i(n+1)$ for some i and each sub-case represents $A_7^i(n)$ for some i. From these cases, we get the following formulas for $A_7^i(n+1)$ in terms of $A_7^i(n)$:

$$\begin{aligned}
A_7^0(n+1) &= 1\cdot A_7^0(n)+0\cdot A_7^1(n)+0\cdot A_7^2(n)+0\cdot A_7^3(n)+0\cdot A_7^4(n)+0\cdot A_7^5(n)+1\cdot A_7^6(n)\\
A_7^1(n+1) &= 0\cdot A_7^0(n)+0\cdot A_7^1(n)+0\cdot A_7^2(n)+0\cdot A_7^3(n)+1\cdot A_7^4(n)+1\cdot A_7^5(n)+0\cdot A_7^6(n)\\
A_7^2(n+1) &= 0\cdot A_7^0(n)+0\cdot A_7^1(n)+1\cdot A_7^2(n)+1\cdot A_7^3(n)+0\cdot A_7^4(n)+0\cdot A_7^5(n)+0\cdot A_7^6(n)\\
A_7^3(n+1) &= 1\cdot A_7^0(n)+1\cdot A_7^1(n)+0\cdot A_7^2(n)+0\cdot A_7^3(n)+0\cdot A_7^4(n)+0\cdot A_7^5(n)+0\cdot A_7^6(n)\\
A_7^4(n+1) &= 0\cdot A_7^0(n)+0\cdot A_7^1(n)+0\cdot A_7^2(n)+0\cdot A_7^3(n)+0\cdot A_7^4(n)+1\cdot A_7^5(n)+1\cdot A_7^6(n)\\
A_7^5(n+1) &= 0\cdot A_7^0(n)+0\cdot A_7^1(n)+0\cdot A_7^2(n)+1\cdot A_7^3(n)+1\cdot A_7^4(n)+0\cdot A_7^5(n)+0\cdot A_7^6(n)\\
A_7^6(n+1) &= 0\cdot A_7^0(n)+1\cdot A_7^1(n)+1\cdot A_7^2(n)+0\cdot A_7^3(n)+0\cdot A_7^4(n)+0\cdot A_7^5(n)+0\cdot A_7^6(n)
\end{aligned}$$

We can rewrite this linear system in terms of matrices. Let

$$A=\begin{bmatrix}1&0&0&0&0&0&1\\0&0&0&0&1&1&0\\0&0&1&1&0&0&0\\1&1&0&0&0&0&0\\0&0&0&0&0&1&1\\0&0&0&1&1&0&0\\0&1&1&0&0&0&0\end{bmatrix},\vec{v}_n=\begin{bmatrix}A_7^0(n)\\A_7^1(n)\\A_7^2(n)\\A_7^3(n)\\A_7^4(n)\\A_7^5(n)\\A_7^6(n)\end{bmatrix}$$

The aforementioned linear system is the same as

$$\vec{v}_{n+1}=\begin{bmatrix}A_7^0(n+1)\\A_7^1(n+1)\\A_7^2(n+1)\\A_7^3(n+1)\\A_7^4(n+1)\\A_7^5(n+1)\\A_7^6(n+1)\end{bmatrix}=\begin{bmatrix}1&0&0&0&0&0&1\\0&0&0&0&1&1&0\\0&0&1&1&0&0&0\\1&1&0&0&0&0&0\\0&0&0&0&0&1&1\\0&0&0&1&1&0&0\\0&1&1&0&0&0&0\end{bmatrix}\begin{bmatrix}A_7^0(n)\\A_7^1(n)\\A_7^2(n)\\A_7^3(n)\\A_7^4(n)\\A_7^5(n)\\A_7^6(n)\end{bmatrix}=A\vec{v}_n$$

By induction, we find that

$$\vec{v}_{n+1}=A\vec{v}_n=A^n\vec{v}_1$$

This linear system allows us to quickly find the number of dreibens of length n that leave a remainder i when divided by 7. We need only calculate the following:

$$\vec{v}_n=A^{n-1}\vec{v}_1$$

Let us investigate A^n to see if we nd anything interesting. Evidently:

$$A=\begin{bmatrix}1&0&0&0&0&0&1\\0&0&0&0&1&1&0\\0&0&1&1&0&0&0\\1&1&0&0&0&0&0\\0&0&0&0&0&1&1\\0&0&0&1&1&0&0\\0&1&1&0&0&0&0\end{bmatrix}=\begin{bmatrix}1&0&0&0&0&0&0\\0&0&0&0&0&1&0\\0&0&0&1&0&0&0\\0&1&0&0&0&0&0\\0&0&0&0&0&0&1\\0&0&0&0&1&0&0\\0&0&1&0&0&0&0\end{bmatrix}+\begin{bmatrix}0&0&0&0&0&0&1\\0&0&0&0&1&0&0\\0&0&1&0&0&0&0\\1&0&0&0&0&0&0\\0&0&0&0&0&1&0\\0&0&0&1&0&0&0\\0&1&0&0&0&0&0\end{bmatrix}$$

Let

$$B=\begin{bmatrix}1&0&0&0&0&0&0\\0&0&0&0&0&1&0\\0&0&0&1&0&0&0\\0&1&0&0&0&0&0\\0&0&0&0&0&0&1\\0&0&0&0&1&0&0\\0&0&1&0&0&0&0\end{bmatrix}\quad\text{and}\quad C=\begin{bmatrix}0&0&0&0&0&0&1\\0&0&0&0&1&0&0\\0&0&1&0&0&0&0\\1&0&0&0&0&0&0\\0&0&0&0&0&1&0\\0&0&0&1&0&0&0\\0&1&0&0&0&0&0\end{bmatrix}$$

so $A=B+C$.

If we look at the powers of B and C, we find that

$$B^i = \begin{cases} \begin{bmatrix} 1&0&0&0&0&0&0\\0&0&0&0&0&1&0\\0&0&0&1&0&0&0\\0&1&0&0&0&0&0\\0&0&0&0&0&0&1\\0&0&0&0&1&0&0\\0&0&1&0&0&0&0 \end{bmatrix} & \text{if } i \equiv 1 \pmod 6 \\ \begin{bmatrix} 1&0&0&0&0&0&0\\0&0&0&0&1&0&0\\0&1&0&0&0&0&0\\0&0&0&0&0&1&0\\0&0&1&0&0&0&0\\0&0&0&0&0&0&1\\0&0&0&1&0&0&0 \end{bmatrix} & \text{if } i \equiv 2 \pmod 6 \\ \begin{bmatrix} 1&0&0&0&0&0&0\\0&0&0&0&0&0&1\\0&0&0&0&0&1&0\\0&0&0&0&1&0&0\\0&0&0&1&0&0&0\\0&0&1&0&0&0&0\\0&1&0&0&0&0&0 \end{bmatrix} & \text{if } i \equiv 3 \pmod 6 \\ \begin{bmatrix} 1&0&0&0&0&0&0\\0&0&1&0&0&0&0\\0&0&0&0&1&0&0\\0&0&0&0&0&0&1\\0&1&0&0&0&0&0\\0&0&0&1&0&0&0\\0&0&0&0&0&1&0 \end{bmatrix} & \text{if } i \equiv 4 \pmod 6 \\ \begin{bmatrix} 1&0&0&0&0&0&0\\0&0&0&1&0&0&0\\0&0&0&0&0&0&1\\0&0&1&0&0&0&0\\0&0&0&0&0&1&0\\0&1&0&0&0&0&0\\0&0&0&0&1&0&0 \end{bmatrix} & \text{if } i \equiv 5 \pmod 6 \\ \begin{bmatrix} 1&0&0&0&0&0&0\\0&1&0&0&0&0&0\\0&0&1&0&0&0&0\\0&0&0&1&0&0&0\\0&0&0&0&1&0&0\\0&0&0&0&0&1&0\\0&0&0&0&0&0&1 \end{bmatrix} & \text{if } i \equiv 0 \pmod 6 \end{cases} \quad \text{and} \quad C^i = \begin{cases} \begin{bmatrix} 0&0&0&0&0&0&1\\0&0&0&0&1&0&0\\0&0&1&0&0&0&0\\1&0&0&0&0&0&0\\0&0&0&0&0&1&0\\0&0&0&1&0&0&0\\0&1&0&0&0&0&0 \end{bmatrix} & \text{if } i \equiv 1 \pmod 6 \\ \begin{bmatrix} 0&1&0&0&0&0&0\\0&0&0&0&0&1&0\\0&0&1&0&0&0&0\\0&0&0&0&0&0&1\\0&0&0&1&0&0&0\\1&0&0&0&0&0&0\\0&0&0&0&1&0&0 \end{bmatrix} & \text{if } i \equiv 2 \pmod 6 \\ \begin{bmatrix} 0&0&0&0&1&0&0\\0&0&0&1&0&0&0\\0&0&1&0&0&0&0\\0&1&0&0&0&0&0\\1&0&0&0&0&0&0\\0&0&0&0&0&0&1\\0&0&0&0&0&1&0 \end{bmatrix} & \text{if } i \equiv 3 \pmod 6 \\ \begin{bmatrix} 0&0&0&0&0&1&0\\1&0&0&0&0&0&0\\0&0&1&0&0&0&0\\0&0&0&0&1&0&0\\0&0&0&0&0&0&1\\0&1&0&0&0&0&0\\0&0&0&1&0&0&0 \end{bmatrix} & \text{if } i \equiv 4 \pmod 6 \\ \begin{bmatrix} 0&0&0&1&0&0&0\\0&0&0&0&0&0&1\\0&0&1&0&0&0&0\\0&0&0&0&0&1&0\\0&1&0&0&0&0&0\\0&0&0&0&1&0&0\\1&0&0&0&0&0&0 \end{bmatrix} & \text{if } i \equiv 5 \pmod 6 \\ \begin{bmatrix} 1&0&0&0&0&0&0\\0&1&0&0&0&0&0\\0&0&1&0&0&0&0\\0&0&0&1&0&0&0\\0&0&0&0&1&0&0\\0&0&0&0&0&1&0\\0&0&0&0&0&0&1 \end{bmatrix} & \text{if } i \equiv 0 \pmod 6 \end{cases}$$

Notice how for $i = 0 \pmod 6$, both B^i and C^i are equal to the identity matrix I_7. Both the powers of B and C are cyclic groups of order 6.

Before examining A^n, let us define the following:

$$\begin{aligned} a(n) &= \sum_{i=0}^{\infty} \binom{n}{6i} = \binom{n}{0} + \binom{n}{6} + \binom{n}{12} + \cdots \\ b(n) &= \sum_{i=0}^{\infty} \binom{n}{6i+1} = \binom{n}{1} + \binom{n}{7} + \binom{n}{13} + \cdots \\ c(n) &= \sum_{i=0}^{\infty} \binom{n}{6i+2} = \binom{n}{2} + \binom{n}{8} + \binom{n}{14} + \cdots \\ d(n) &= \sum_{i=0}^{\infty} \binom{n}{6i+3} = \binom{n}{3} + \binom{n}{9} + \binom{n}{15} + \cdots \\ e(n) &= \sum_{i=0}^{\infty} \binom{n}{6i+4} = \binom{n}{4} + \binom{n}{10} + \binom{n}{16} + \cdots \\ f(n) &= \sum_{i=0}^{\infty} \binom{n}{6i+5} = \binom{n}{5} + \binom{n}{11} + \binom{n}{17} + \cdots \end{aligned}$$

These formulas will be useful for condensing our notation.

Now we can finally begin examining A^n:

$$\begin{aligned} A &= B + C \\ A^n &= (B+C)^n \\ &= \binom{n}{0}B^n + \binom{n}{1}B^{n-1}C + \binom{n}{2}B^{n-2}C^2 + \cdots + \binom{n}{n-2}B^2C^{n-2} + \binom{n}{n-1}BC^{n-1} + \binom{n}{n}C^n \\ &= \left[\binom{n}{0}B^n + \binom{n}{6}B^{n-6}C^6 + \binom{n}{12}B^{n-12}C^{12} + \cdots\right] + \left[\binom{n}{1}B^{n-1}C + \binom{n}{7}B^{n-7}C^7 + \binom{n}{13}B^{n-13}C^{13} + \cdots\right] \\ &\quad + \left[\binom{n}{2}B^{n-2}C^2 + \binom{n}{8}B^{n-8}C^8 + \binom{n}{14}B^{n-14}C^{14} + \cdots\right] + \left[\binom{n}{3}B^{n-3}C^3 + \binom{n}{9}B^{n-9}C^9 + \binom{n}{15}B^{n-15}C^{15} + \cdots\right] \\ &\quad + \left[\binom{n}{4}B^{n-4}C^4 + \binom{n}{10}B^{n-10}C^{10} + \binom{n}{16}B^{n-16}C^{16} + \cdots\right] + \left[\binom{n}{5}B^{n-5}C^5 + \binom{n}{11}B^{n-11}C^{11} + \binom{n}{17}B^{n-17}C^{17} + \cdots\right] \\ &= \left[\binom{n}{0}B^n + \binom{n}{6}B^n + \binom{n}{12}B^n + \cdots\right] + \left[\binom{n}{1}B^{n-1}C + \binom{n}{7}B^{n-1}C + \binom{n}{13}B^{n-1}C + \cdots\right] \\ &\quad + \left[\binom{n}{2}B^{n-2}C^2 + \binom{n}{8}B^{n-2}C^2 + \binom{n}{14}B^{n-2}C^2 + \cdots\right] + \left[\binom{n}{3}B^{n-3}C^3 + \binom{n}{9}B^{n-3}C^3 + \binom{n}{15}B^{n-3}C^3 + \cdots\right] \\ &\quad + \left[\binom{n}{4}B^{n-4}C^4 + \binom{n}{10}B^{n-4}C^4 + \binom{n}{16}B^{n-4}C^4 + \cdots\right] + \left[\binom{n}{5}B^{n-5}C^5 + \binom{n}{11}B^{n-5}C^5 + \binom{n}{17}B^{n-5}C^5 + \cdots\right] \\ &= \left[\binom{n}{0} + \binom{n}{6} + \binom{n}{12} + \cdots\right]B^n + \left[\binom{n}{1} + \binom{n}{7} + \binom{n}{13} + \cdots\right]B^{n-1}C + \left[\binom{n}{2} + \binom{n}{8} + \binom{n}{14} + \cdots\right]B^{n-2}C^2 \\ &\quad + \left[\binom{n}{3} + \binom{n}{9} + \binom{n}{15} + \cdots\right]B^{n-3}C^3 + \left[\binom{n}{4} + \binom{n}{10} + \binom{n}{16} + \cdots\right]B^{n-4}C^4 + \left[\binom{n}{5} + \binom{n}{11} + \binom{n}{17} + \cdots\right]B^{n-5}C^5 \\ &= a(n)B^n + b(n)B^{n-1}C + c(n)B^{n-2}C^2 + d(n)B^{n-3}C^3 + e(n)B^{n-4}C^4 + f(n)B^{n-5}C^5 \end{aligned}$$

Now we have

$$A^n = a(n)B^n + b(n)B^{n-1}C + c(n)B^{n-2}C^2 + d(n)B^{n-3}C^3 + e(n)B^{n-4}C^4 + f(n)B^{n-5}C^5$$

The problem with the above equation is that we still have the powers of B in terms of n. However, we can easily solve this problem because the powers of B form a cyclic group. Since this cyclic group has order 6, we must consider six cases.

Case 1: $n \equiv 0 \pmod 6$

$$\begin{aligned} A^n &= a(n)I + b(n)B^5C + c(n)B^4C^2 + d(n)B^3C^3 + e(n)B^2C^4 + f(n)BC^5 \\ &= \begin{bmatrix} a(n) & c(n) & 0 & f(n) & d(n) & e(n) & b(n) \\ b(n) & a(n) & c(n) & 0 & f(n) & d(n) & e(n) \\ e(n) & b(n) & a(n) & c(n) & 0 & f(n) & d(n) \\ d(n) & e(n) & b(n) & a(n) & c(n) & 0 & f(n) \\ f(n) & d(n) & e(n) & b(n) & a(n) & c(n) & 0 \\ 0 & f(n) & d(n) & e(n) & b(n) & a(n) & c(n) \\ c(n) & 0 & f(n) & d(n) & e(n) & b(n) & a(n) \end{bmatrix} \end{aligned}$$

Case 2: $n \equiv 1 \pmod 6$

$$\begin{aligned} A^n &= a(n)B + b(n)C + c(n)B^5C^2 + d(n)B^4C^3 + e(n)B^3C^4 + f(n)B2C^5 \\ &= \begin{bmatrix} a(n) & c(n) & 0 & f(n) & d(n) & e(n) & b(n) \\ 0 & f(n) & d(n) & e(n) & b(n) & a(n) & c(n) \\ d(n) & e(n) & b(n) & a(n) & c(n) & 0 & f(n) \\ b(n) & a(n) & c(n) & 0 & f(n) & d(n) & e(n) \\ c(n) & 0 & f(n) & d(n) & e(n) & b(n) & a(n) \\ f(n) & d(n) & e(n) & b(n) & a(n) & c(n) & 0 \\ e(n) & b(n) & a(n) & c(n) & 0 & f(n) & d(n) \end{bmatrix} \end{aligned}$$

Case 3: $n \equiv 2 \pmod 6$

$$\begin{aligned} A^n &= a(n)B^2 + b(n)BC + c(n)C^2 + d(n)B^5C^3 + e(n)B^4C^4 + f(n)B3C^5 \\ &= \begin{bmatrix} a(n) & c(n) & 0 & f(n) & d(n) & e(n) & b(n) \\ f(n) & d(n) & e(n) & b(n) & a(n) & c(n) & 0 \\ b(n) & a(n) & c(n) & 0 & f(n) & d(n) & e(n) \\ 0 & f(n) & d(n) & e(n) & b(n) & a(n) & c(n) \\ e(n) & b(n) & a(n) & c(n) & 0 & f(n) & d(n) \\ c(n) & 0 & f(n) & d(n) & e(n) & b(n) & a(n) \\ d(n) & e(n) & b(n) & a(n) & c(n) & 0 & f(n) \end{bmatrix} \end{aligned}$$

Case 4: $n \equiv 3 \pmod 6$

$$A^n = a(n)B^3 + b(n)B^2C + c(n)BC^2 + d(n)C^3 + e(n)B^5C^4 + f(n)B4C^5$$
$$= \begin{bmatrix} a(n) & c(n) & 0 & f(n) & d(n) & e(n) & b(n) \\ c(n) & 0 & f(n) & d(n) & e(n) & b(n) & a(n) \\ 0 & f(n) & d(n) & e(n) & b(n) & a(n) & c(n) \\ f(n) & d(n) & e(n) & b(n) & a(n) & c(n) & 0 \\ d(n) & e(n) & b(n) & a(n) & c(n) & 0 & f(n) \\ e(n) & b(n) & a(n) & c(n) & 0 & f(n) & d(n) \\ b(n) & a(n) & c(n) & 0 & f(n) & d(n) & e(n) \end{bmatrix}$$

Case 5: $n \equiv 4 \pmod 6$

$$A^n = a(n)B^4 + b(n)B^3C + c(n)B^2C^2 + d(n)BC^3 + e(n)C^4 + f(n)B5C^5$$
$$= \begin{bmatrix} a(n) & c(n) & 0 & f(n) & d(n) & e(n) & b(n) \\ e(n) & b(n) & a(n) & c(n) & 0 & f(n) & d(n) \\ f(n) & d(n) & e(n) & b(n) & a(n) & c(n) & 0 \\ c(n) & 0 & f(n) & d(n) & e(n) & b(n) & a(n) \\ b(n) & a(n) & c(n) & 0 & f(n) & d(n) & e(n) \\ d(n) & e(n) & b(n) & a(n) & c(n) & 0 & f(n) \\ 0 & f(n) & d(n) & e(n) & b(n) & a(n) & c(n) \end{bmatrix}$$

Case 6: $n \equiv 5 \pmod 6$

$$A^n = a(n)B^5 + b(n)B^4C + c(n)B^3C^2 + d(n)B^2C^3 + e(n)B^1C^4 + f(n)C^5$$
$$= \begin{bmatrix} a(n) & c(n) & 0 & f(n) & d(n) & e(n) & b(n) \\ d(n) & e(n) & b(n) & a(n) & c(n) & 0 & f(n) \\ c(n) & 0 & f(n) & d(n) & e(n) & b(n) & a(n) \\ e(n) & b(n) & a(n) & c(n) & 0 & f(n) & d(n) \\ 0 & f(n) & d(n) & e(n) & b(n) & a(n) & c(n) \\ b(n) & a(n) & c(n) & 0 & f(n) & d(n) & e(n) \\ f(n) & d(n) & e(n) & b(n) & a(n) & c(n) & 0 \end{bmatrix}$$

Evidently, no matter what n is, the first row of the matrix A^n is always

$$\begin{bmatrix} a(n) & c(n) & 0 & f(n) & d(n) & e(n) & b(n) \end{bmatrix}$$

Recall that

$$\vec{v}_{n+1} = A^n \vec{v}_1$$

If we want to find $A^0_7(n+1)$, i.e. the first entry of $\vec{v}_{n+1}$, we need only consider the first row of A^n:

$$\begin{aligned} A^0_7(n+1) &= a(n)A^0_7(1) + c(n)A^1_7(1) + 0A^2_7(1) + f(n)A^3_7(1) + d(n)A^4_7(1) + e(n)A^5_7(1) + b(n)A^6_7(1) \\ &= a(n)(1) + c(n)(0) + 0(0) + f(n)(1) + d(n)(0) + e(n)(0) + b(n)(0) \\ &= a(n) + f(n) \\ A^0_7(n) &= a(n-1) + f(n-1) \\ &= \left[\binom{n-1}{0} + \binom{n-1}{6} + \binom{n-1}{12} + \cdots\right] + \left[\binom{n-1}{5} + \binom{n-1}{11} + \binom{n-1}{17} + \cdots\right] \\ &= \sum_{i=0}^{\infty} \binom{n-1}{6i} + \sum_{i=0}^{\infty} \binom{n-1}{6i+5} \end{aligned}$$

Now we have a formula with which to find the number of dreibens of length n that are divisible by 7:

$$A^0_7(n) = a(n-1) + f(n-1)$$

What if we want to find any $A^i_7(n)$, $i = 0, 1, 2, 3, 4, 5, 6$? We know that these $A^i_7(n)$ are entries of $\vec{v}_n$. Recall that

$$\vec{v}_{n+1} = A^n \vec{v}_1$$

To solve for $\vec{v}_{n+1}$, we must first understand A^n. Recall that

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

As it turns out, A is a diagonalizable matrix, so A^n is also diagonalizable:

$$A^n = PD^nP^{-1}$$

$$P = \begin{bmatrix} 0 & -2 & 1 & \frac{1}{6}i\left(3i+\sqrt{3}\right) & \frac{1}{2}i\left(i+\sqrt{3}\right) & -\frac{1}{6}i\left(-3i+\sqrt{3}\right) & -\frac{1}{2}i\left(-i+\sqrt{3}\right) \\ 0 & 2 & 1 & \frac{-2i}{\sqrt{3}} & 0 & \frac{2i}{\sqrt{3}} & 0 \\ 0 & -2 & 1 & \frac{1}{6}i\left(3i+\sqrt{3}\right) & \frac{1}{2}\left(1-i\sqrt{3}\right) & \frac{-1}{6}i\left(-3i+\sqrt{3}\right) & \frac{1}{2}\left(1+i\sqrt{3}\right) \\ 0 & 0 & 1 & 1 & -1 & 1 & -1 \\ -1 & 1 & 1 & \frac{1}{6}i\left(3i+\sqrt{3}\right) & \frac{1}{6}\left(3+i\sqrt{3}\right) & -\frac{1}{6}\left(-3i+\sqrt{3}\right) & \frac{1}{6}\left(3-i\sqrt{3}\right) \\ 1 & 1 & 1 & \frac{1}{6}i\left(3i+\sqrt{3}\right) & -\frac{1}{6}i\left(-3i+\sqrt{3}\right) & -\frac{1}{6}\left(-3i+\sqrt{3}\right) & \frac{1}{6}\left(3i+\sqrt{3}\right) \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2}i\left(-i+\sqrt{3}\right) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2}\left(1-i\sqrt{3}\right) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}i\left(i+\sqrt{3}\right) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2}\left(1+i\sqrt{3}\right) \end{bmatrix}$$

$$P^{-1} = \begin{bmatrix} \frac{1}{6} & 0 & -\frac{1}{6} & -\frac{1}{3} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{3} \\ -\frac{1}{6} & 0 & -\frac{1}{6} & 0 & \frac{1}{6} & \frac{1}{6} & 0 \\ \frac{1}{7} & \frac{1}{7} & \frac{1}{7} & \frac{1}{7} & \frac{1}{7} & \frac{1}{7} & \frac{1}{7} \\ \frac{1}{42}i\left(3i+2\sqrt{3}\right) & \frac{1}{14}i\left(i+3\sqrt{3}\right) & \frac{1}{42}i\left(3i+2\sqrt{3}\right) & \frac{1}{28}\left(5-i\sqrt{3}\right) & -\frac{1}{42}i\left(-3i+5\sqrt{3}\right) & -\frac{1}{42}i\left(-3i+5\sqrt{3}\right) & \frac{1}{28}\left(5-i\sqrt{3}\right) \\ -\frac{i}{2\sqrt{3}} & 0 & \frac{i}{2\sqrt{3}} & \frac{1}{12}i\left(3i+\sqrt{3}\right) & 0 & 0 & \frac{1}{12}\left(3-i\sqrt{3}\right) \\ \frac{1}{42}\left(-3-2i\sqrt{3}\right) & \frac{1}{14}\left(-1-3i\sqrt{3}\right) & \frac{1}{42}\left(-3-2i\sqrt{3}\right) & \frac{1}{28}\left(5+i\sqrt{3}\right) & \frac{1}{42}i\left(3i+5\sqrt{3}\right) & \frac{1}{42}i\left(3i+5\sqrt{3}\right) & \frac{1}{28}\left(5+i\sqrt{3}\right) \\ \frac{i}{2\sqrt{3}} & 0 & -\frac{i}{2\sqrt{3}} & -\frac{1}{12}i\left(-3i+\sqrt{3}\right) & 0 & 0 & \frac{1}{12}\left(3+i\sqrt{3}\right) \end{bmatrix}$$

Interestingly, some entries of D are 6th roots of unity. If we let ζ_i^j be the jth ith-root of unity, then

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2}i\left(-i+\sqrt{3}\right) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2}\left(1-i\sqrt{3}\right) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}i\left(i+\sqrt{3}\right) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2}\left(1+i\sqrt{3}\right) \end{bmatrix}$$
$$= \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2}-i\frac{\sqrt{3}}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2}-i\frac{\sqrt{3}}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{2}+i\frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2}+i\frac{\sqrt{3}}{2} \end{bmatrix}$$
$$= \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{\frac{4\pi i}{3}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{\frac{5\pi i}{3}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & e^{\frac{2\pi i}{3}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & e^{\frac{\pi i}{3}} \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \zeta_6^4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \zeta_6^5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \zeta_6^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \zeta_6^1 \end{bmatrix}$$

$$D^n = \begin{bmatrix} (-1)^n & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2^n & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \zeta_6^{4n} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \zeta_6^{5n} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \zeta_6^{2n} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \zeta_6^{n} \end{bmatrix}$$

Recall that the nth roots of unity form a cyclic group of order n, as in Table 1 below:

Table 1: Powers of the 6th roots of unity

$n \pmod 6$	ζ_6^{0n}	ζ_6^{n}	ζ_6^{2n}	ζ_6^{3n}	ζ_6^{4n}	ζ_6^{5n}
0	1	1	1	1	1	1
1	1	ζ_6^1	ζ_6^2	-1	ζ_6^4	ζ_6^5
2	1	ζ_6^2	ζ_6^4	1	ζ_6^2	ζ_6^4
3	1	-1	1	-1	1	-1
4	1	ζ_6^4	ζ_6^2	1	ζ_6^4	ζ_6^2
5	1	ζ_6^5	ζ_6^4	-1	ζ_6^2	ζ_6^1

Based on the properties of the 6th roots of unity, we find that

$$D^n = \begin{cases} \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2^n & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \zeta_6^4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \zeta_6^5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \zeta_6^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \zeta_6^1 \end{bmatrix} & \text{if } n \equiv 1 \pmod 6 \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2^n & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \zeta_6^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \zeta_6^4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \zeta_6^4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \zeta_6^2 \end{bmatrix} & \text{if } n \equiv 2 \pmod 6 \\ \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2^n & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix} & \text{if } n \equiv 3 \pmod 6 \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2^n & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \zeta_6^4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \zeta_6^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \zeta_6^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \zeta_6^4 \end{bmatrix} & \text{if } n \equiv 4 \pmod 6 \\ \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2^n & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \zeta_6^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \zeta_6^1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \zeta_6^4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \zeta_6^5 \end{bmatrix} & \text{if } n \equiv 5 \pmod 6 \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2^n & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} & \text{if } n \equiv 0 \pmod 6 \end{cases}$$

Now that we understand D^n, we are much closer to characterizing A^n. All we need to do is to calculate PD^nP^{-1} for each case of D^n. Below, we will calculate the first case, i.e. where $n \equiv 0 \pmod 6$.

If $n \equiv 0 \pmod 6$,

If $n \equiv 0 \pmod 6$,

$$D^n = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2^n & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$PD^n = \begin{bmatrix} 0 & -2 & 2^n & -\frac{1}{2} + i\frac{\sqrt{3}}{6} & \zeta_6^2 & -\frac{1}{2} - i\frac{\sqrt{3}}{6} & \zeta_6^4 \\ 0 & 2 & 2^n & -i\frac{2}{\sqrt{3}} & 0 & i\frac{2}{\sqrt{3}} & 0 \\ 0 & -2 & 2^n & -\frac{1}{2} + i\frac{\sqrt{3}}{6} & \zeta_6^5 & -\frac{1}{2} - i\frac{\sqrt{3}}{6} & \zeta_6^1 \\ 0 & 0 & 2^n & 1 & -1 & 1 & -1 \\ -1 & 1 & 2^n & -\frac{1}{2} + i\frac{\sqrt{3}}{6} & \frac{1}{2} + i\frac{\sqrt{3}}{6} & -\frac{1}{2} - i\frac{\sqrt{3}}{6} & \frac{1}{2} - i\frac{\sqrt{3}}{6} \\ 1 & 1 & 2^n & -\frac{1}{2} + i\frac{\sqrt{3}}{6} & -\frac{1}{2} - i\frac{\sqrt{3}}{6} & -\frac{1}{2} - i\frac{\sqrt{3}}{6} & -\frac{1}{2} + i\frac{\sqrt{3}}{6} \\ 0 & 0 & 2^n & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$A^n = PD^nP^{-1} = \begin{bmatrix} \frac{1}{7}(2^n+6) & \frac{1}{7}(2^n-1) & \frac{1}{7}(2^n-1) & \frac{1}{7}(2^n-1) & \frac{1}{7}(2^n-1) & \frac{1}{7}(2^n-1) & \frac{1}{7}(2^n-1) \\ \frac{1}{7}(2^n-1) & \frac{1}{7}(2^n+6) & \frac{1}{7}(2^n-1) & \frac{1}{7}(2^n-1) & \frac{1}{7}(2^n-1) & \frac{1}{7}(2^n-1) & \frac{1}{7}(2^n-1) \\ \frac{1}{7}(2^n-1) & \frac{1}{7}(2^n-1) & \frac{1}{7}(2^n+6) & \frac{1}{7}(2^n-1) & \frac{1}{7}(2^n-1) & \frac{1}{7}(2^n-1) & \frac{1}{7}(2^n-1) \\ \frac{1}{7}(2^n-1) & \frac{1}{7}(2^n-1) & \frac{1}{7}(2^n-1) & \frac{1}{7}(2^n+6) & \frac{1}{7}(2^n-1) & \frac{1}{7}(2^n-1) & \frac{1}{7}(2^n-1) \\ \frac{1}{7}(2^n-1) & \frac{1}{7}(2^n-1) & \frac{1}{7}(2^n-1) & \frac{1}{7}(2^n-1) & \frac{1}{7}(2^n+6) & \frac{1}{7}(2^n-1) & \frac{1}{7}(2^n-1) \\ \frac{1}{7}(2^n-1) & \frac{1}{7}(2^n-1) & \frac{1}{7}(2^n-1) & \frac{1}{7}(2^n-1) & \frac{1}{7}(2^n-1) & \frac{1}{7}(2^n+6) & \frac{1}{7}(2^n-1) \\ \frac{1}{7}(2^n-1) & \frac{1}{7}(2^n-1) & \frac{1}{7}(2^n-1) & \frac{1}{7}(2^n-1) & \frac{1}{7}(2^n-1) & \frac{1}{7}(2^n-1) & \frac{1}{7}(2^n+6) \end{bmatrix}$$

Where $n \equiv 0 \pmod 6$, the entries of A^n are all either $1/7(2^n - 1)$ or $1/7(2^n + 6)$. Hence, A^n can be much more easily calculated; instead of performing lengthy matrix power operations, one can merely find $1/7(2n - 1)$ and $1/7(2n + 6)$ to determine A^n. Since $\vec{v}n = A^{n-1}\vec{v}1$, one can now easily determine any $A^i_7(n)$ where $n \equiv 0 \pmod 6$. In other words, one can easily determine the number of dreibens of length $n \equiv 0 \pmod 6$ that leave remainder i when divided by 7. By extension, one can determine the number of dreibens of length $n \equiv 0 \pmod 6$ divisible by 7; all of these dreibens are necessarily composite, so they can be ruled out during our search for infinitely many prime dreibens.

We can perform similar calculations for A^n where $n \equiv 1, 2, 3, 4, 5 \pmod 6$, thereby allowing us to quickly find A^n for any n. Hence, we can easily determine A^n for any n, allowing us to rule out many composite dreibens.

COMBINATORIAL APPROACH TO DREIBENS MODULO 7

Let us take a combinatorial approach to $A^i_7(n)$. For now, let us restrict ourselves to $A^0_7(n)$; in other words, we are looking for when a dreiben of length n is divisible by 7. These dreibens are obviously composite, so learning to identify them will help rule out composite dreibens in a prime dreiben search.

Here, we will use the divisibility rule for 7 to aid us. Recall the divisibility rule for 7:

$$10^x \equiv \begin{cases} 1 \pmod 7 & \text{if } x \equiv 0 \pmod 6 \\ 3 \pmod 7 & \text{if } x \equiv 1 \pmod 6 \\ 2 \pmod 7 & \text{if } x \equiv 2 \pmod 6 \\ 6 \pmod 7 & \text{if } x \equiv 3 \pmod 6 \\ 4 \pmod 7 & \text{if } x \equiv 4 \pmod 6 \\ 5 \pmod 7 & \text{if } x \equiv 5 \pmod 6 \end{cases}$$

Notice that the powers of 10 (mod 7) forms a cyclic group of order 6.

Let us use the above divisibility rule to determine, as an example, 9487310294 (mod 7):

$$\begin{aligned} 9487310294 &= 9\,(10^9) + 4\,(10^8) + 8\,(10^7) + 7\,(10^6) + 3\,(10^5) + 1\,(10^4) + 0\,(10^3) + 2\,(10^2) + (10)\,9 + 4 \\ &\equiv 9\,(10^3) + 4\,(10^2) + 8\,(10) + 7 + 3\,(10^5) + 1\,(10^4) + 0\,(10^3) + 2\,(10^2) + (10)\,9 + 4 \\ &\equiv 9\,(6) + 4\,(2) + 8\,(3) + 7 + 3\,(5) + 1\,(4) + 0\,(6) + 2\,(2) + (3)\,9 + 4 \\ &\equiv 54 + 8 + 24 + 0 + 15 + 4 + 0 + 4 + 27 + 4 \\ &\equiv 140 \equiv 0 \pmod 7 \end{aligned}$$

Dreibens consist of only 7's and 3's; hence, the divisibility of a dreiben by 7 depends on both the number and placement of 3's amongst its digits. (Neither the number nor placement of 7's in a dreiben affect its divisibility by 7, as we can tell from the divisibility rule for 7.) Let $D^n_j(i)$ be the number of dreibens of length n with i number of j's in it, e.g. $D^n_3(2)$ is the number of dreibens of length n where 3 occupies exactly 2 digits of each dreiben. Since the divisibility of a dreiben by 7 depends on the number and placement of 3's amongst that dreiben's digits, we would stand to gain from investigating $D^n_3(i)$.

As you may have realized already,

$$A^0_7(n) = \sum_{k=0}^{\infty} D^n_3(k) = D^n_3(0) + D^n_3(1) + D^n_3(2) + D^n_3(3) + \cdots$$

Evidently, by examining $D^n_3(i)$, we can approach the problem of determining the number of dreibens of length n divisible by 7. Our approach will involve investigating $D^n_3(i)$ with varying i to see if we can make any generalizations about $D^n_3(i)$.

First, let us examine $D^n_3(0)$. There is only one way to put exactly zero 3's in a dreiben; every digit of that dreiben must be 7. Obviously, if a number's digits are all 7's, then the number is divisible by 7. Since all of the digits of a dreiben with zero 3's is 7, and since there is only one way to make a dreiben of length n with zero 3's, we can easily see that $D^n_3(0) = 1$.

Next, we can examine $D^n_3(1) = 0$. There are no dreibens with only one 3, since 3 times any digit place 10^x is never divisible by 7, so $D^n_3(1) = 0$.

So far, we have covered two very simple instances of $D^n_3(i)$. When $i = 0$, there is only a single way to make a dreiben divisible by 7. When $i = 1$, it is impossible to make a dreiben divisible by 7. However, as i increases, the problem of describing $D^n_3(i)$ becomes much more complicated. The next question in our investigation of $D^n_3(i)$ involves $i = 2$. In other words, how many ways are there to put exactly two 3's in a dreiben of length n? Putting exactly two 3's in a dreiben of length n means we must choose two digit places 10^x and 10^y where $x \neq y$ and $0 \leq x, y \leq n - 1$. Putting our two 3's in these two digit places, we find that the dreiben is congruent to $3(10^x + 10^y) \pmod 7$. Clearly, the dreiben is divisible by 7 if and only if $10^x + 10^y$ is divisible by 7, so we need only consider $10^x + 10^y \pmod 7$.

What are the possible ways to make $10^x + 10^y \equiv 0 \pmod 7$? The powers of 10 modulo 7 form a cyclic group of order 6; without loss of generality, the possible solutions are

(a) $10^x \equiv 1 \pmod 7$, $10^y \equiv 6 \pmod 7$

(b) $10^x \equiv 2 \pmod 7$, $10^y \equiv 5 \pmod 7$

(c) $10^x \equiv 3 \pmod 7$, $10^y \equiv 4 \pmod 7$

Another way to write the possible solutions is

(a) $x \equiv 0 \pmod 6$, $y \equiv 3 \pmod 6$

(b) $x \equiv 2 \pmod 6$, $y \equiv 5 \pmod 6$

(c) $x \equiv 1 \pmod 6$, $y \equiv 4 \pmod 6$

For each solution, we choose one digit place each from two sets of digit places. Here, we introduce some notation that will make the later calculations more concise:

- α is the set of digit places $10^x \equiv 1 \pmod 7$, i.e. $x \equiv 0 \pmod 6$.
- β is the set of digit places $10^x \equiv 2 \pmod 7$, i.e. $x \equiv 2 \pmod 6$.
- γ is the set of digit places $10^x \equiv 3 \pmod 7$, i.e. $x \equiv 1 \pmod 6$.
- δ is the set of digit places $10^x \equiv 4 \pmod 7$, i.e. $x \equiv 4 \pmod 6$.
- ε is the set of digit places $10^x \equiv 5 \pmod 7$, i.e. $x \equiv 5 \pmod 6$.
- ζ is the set of digit places $10^x \equiv 6 \pmod 7$, i.e. $x \equiv 3 \pmod 6$.

As an example, one possible way to put two 3's into a dreiben is to choose one digit place from α and another digit place from ζ. The number of possible combinations of choosing two digit places, one from α and one from ζ, is $\binom{|\alpha|}{1}\binom{|\zeta|}{1} = |\alpha| \cdot |\zeta|$. Applying this to the other sets of digit places, we find that

$$D_3^n(2) = |\alpha| \cdot |\zeta| + |\beta| \cdot |\epsilon| + |\gamma| \cdot |\delta|$$

As you may have noticed, the cardinality of the six sets above depends on $n \pmod 6$. This can be seen in Table 2 below:

Table 2: Cardinality of the sets α through ζ

$n \pmod 6$	$\lvert\alpha\rvert$	$\lvert\beta\rvert$	$\lvert\gamma\rvert$	$\lvert\delta\rvert$	$\lvert\epsilon\rvert$	$\lvert\zeta\rvert$
0	$\frac{n}{6}$	$\frac{n}{6}$	$\frac{n}{6}$	$\frac{n}{6}$	$\frac{n}{6}$	$\frac{n}{6}$
1	$\frac{n+5}{6}$	$\frac{n-1}{6}$	$\frac{n-1}{6}$	$\frac{n-1}{6}$	$\frac{n-1}{6}$	$\frac{n-1}{6}$
2	$\frac{n+4}{6}$	$\frac{n-2}{6}$	$\frac{n+4}{6}$	$\frac{n-2}{6}$	$\frac{n-2}{6}$	$\frac{n-2}{6}$
3	$\frac{n+3}{6}$	$\frac{n+3}{6}$	$\frac{n+3}{6}$	$\frac{n-3}{6}$	$\frac{n-3}{6}$	$\frac{n-3}{6}$
4	$\frac{n+2}{6}$	$\frac{n+2}{6}$	$\frac{n+2}{6}$	$\frac{n-4}{6}$	$\frac{n-4}{6}$	$\frac{n+2}{6}$
5	$\frac{n+1}{6}$	$\frac{n+1}{6}$	$\frac{n+1}{6}$	$\frac{n+1}{6}$	$\frac{n-5}{6}$	$\frac{n+1}{6}$

Hence,

- If $n \equiv 0 \pmod 6$, then

$$D_3^n(2) = \frac{n}{6} \cdot \frac{n}{6} + \frac{n}{6} \cdot \frac{n}{6} + \frac{n}{6} \cdot \frac{n}{6} = \frac{3n^2}{36} = \frac{n^2}{12}$$

- If $n \equiv 1 \pmod 6$, then

$$D_3^n(2) = \frac{n+5}{6} \cdot \frac{n-1}{6} + \frac{n-1}{6} \cdot \frac{n-1}{6} + \frac{n-1}{6} \cdot \frac{n-1}{6} = \frac{(n+5)(n-1) + 2(n-1)^2}{36}$$
$$= \frac{(n^2+4n-5) + 2(n^2-2n+1)}{36} = \frac{3n^2-3}{36} = \frac{n^2-1}{12}$$

- If $n \equiv 2 \pmod 6$, then

$$D_3^n(2) = \frac{n+4}{6} \cdot \frac{n-2}{6} + \frac{n-2}{6} \cdot \frac{n-2}{6} + \frac{n+4}{6} \cdot \frac{n-2}{6} = \frac{2(n+4)(n-2) + (n-2)^2}{36}$$
$$= \frac{2(n^2+2n-8) + (n^2-4n+4)}{36} = \frac{3n^2-12}{36} = \frac{n^2-4}{12}$$

- If $n \equiv 3 \pmod 6$, then

$$D_3^n(2) = \frac{n+3}{36} \cdot \frac{n-3}{36} + \frac{n+3}{36} \cdot \frac{n-3}{36} + \frac{n+3}{36} \cdot \frac{n-3}{36} = \frac{3(n+3)(n-3)}{36}$$
$$= \frac{3(n^2-9)}{36} = \frac{n^2-9}{12}$$

- If $n \equiv 4 \pmod 6$, then

$$D_3^n(2) = \frac{n+2}{36} \cdot \frac{n+2}{36} + \frac{n+2}{36} \cdot \frac{n-4}{36} + \frac{n+2}{36} \cdot \frac{n-4}{36} = \frac{(n^2+4n+4) + 2(n^2-2n-8)}{36}$$
$$= \frac{3n^2-12}{36} = \frac{n^2-4}{12}$$

- If $n \equiv 5 \pmod 6$, then

$$D_3^n(2) = \frac{n+1}{36} \cdot \frac{n+1}{36} + \frac{n+1}{36} \cdot \frac{n+1}{36} + \frac{n+1}{36} \cdot \frac{n-5}{36} = \frac{2(n+1)^2 + (n+1)(n-5)}{36}$$
$$= \frac{2(n^2+2n+1) + (n^2-4n-5)}{36} = \frac{3n^2-3}{36} = \frac{n^2-1}{12}$$

Evidently, $D_3^n(2) = [\, n^2/12 \,]$. Let us prove this.

Proof. If $n \equiv 0 \pmod 6$, i.e. $n = 6m$ where m is an integer, then

$$D_3^n(2) = \frac{n^2}{12} = \frac{(6m)^2}{12} = \frac{36m^2}{12} = 3m^2$$
$$\left\lfloor \frac{n^2}{12} \right\rfloor = \left\lfloor \frac{(6m)^2}{12} \right\rfloor = \lfloor 3m^2 \rfloor = 3m^2$$
$$D_3^n(2) = \left\lfloor \frac{n^2}{12} \right\rfloor$$

If $n \equiv 1 \pmod 6$, i.e. $n = 6m + 1$ where m is an integer, then

$$D_3^n(2) = \frac{n^2-1}{12} = \frac{(6m+1)^2-1}{12} = \frac{36m^2+12m+1-1}{12} = 3m^2+m$$
$$\left\lfloor \frac{n^2}{12} \right\rfloor = \left\lfloor \frac{(6m+1)^2}{12} \right\rfloor = \left\lfloor \frac{36m^2+12m+1}{12} \right\rfloor = \left\lfloor 3m^2+m+\frac{1}{12} \right\rfloor = 3m^2+m$$
$$D_3^n(2) = \left\lfloor \frac{n^2}{12} \right\rfloor$$

If $n \equiv 2 \pmod 6$, i.e. $n = 6m + 2$ where m is an integer, then

$$D_3^n(2) = \frac{n^2-4}{12} = \frac{(6m+2)^2-4}{12} = \frac{36m^2+24m+4-4}{12} = 3m^2+2m$$
$$\left\lfloor \frac{n^2}{12} \right\rfloor = \left\lfloor \frac{(6m+2)^2}{12} \right\rfloor = \left\lfloor \frac{36m^2+24m+4}{12} \right\rfloor = \left\lfloor 3m^2+2m+\frac{1}{3} \right\rfloor = 3m^2+2m$$
$$D_3^n(2) = \left\lfloor \frac{n^2}{12} \right\rfloor$$

If $n \equiv 3 \pmod 6$, i.e. $n = 6m + 3$ where m is an integer, then

$$D_3^n(2) = \frac{n^2-9}{12} = \frac{(6m+3)^2-9}{12} = \frac{36m^2+36m+9-9}{12} = 3m^2+3m$$
$$\left\lfloor \frac{n^2}{12} \right\rfloor = \left\lfloor \frac{(6m+3)^2}{12} \right\rfloor = \left\lfloor \frac{36m^2+36m+9}{12} \right\rfloor = \left\lfloor 3m^2+3m+\frac{3}{4} \right\rfloor = 3m^2+3m$$
$$D_3^n(2) = \left\lfloor \frac{n^2}{12} \right\rfloor$$

If $n \equiv 4 \pmod 6$, i.e. $n = 6m + 4$ where m is an integer, then

$$D_3^n(2) = \frac{n^2-4}{12} = \frac{(6m+4)^2-4}{12} = \frac{36m^2+48m+16-4}{12} = \frac{36m^2+48m+12}{12} = 3m^2+4m+1$$
$$\left\lfloor \frac{n^2}{12} \right\rfloor = \left\lfloor \frac{(6m+4)^2}{12} \right\rfloor = \left\lfloor \frac{36m^2+48m+16}{12} \right\rfloor = \left\lfloor 3m^2+4m+1+\frac{1}{3} \right\rfloor = 3m^2+4m+1$$
$$D_3^n(2) = \left\lfloor \frac{n^2}{12} \right\rfloor$$

If $n \equiv 5 \pmod 6$, i.e. $n = 6m + 5$ where m is an integer, then

$$D_3^n(2) = \frac{n^2-1}{12} = \frac{(6m+5)^2-1}{12} = \frac{36m^2+60m+25-1}{12} = \frac{36m^2+60m+24}{12} = 3m^2+5m+2$$
$$\left\lfloor \frac{n^2}{12} \right\rfloor = \left\lfloor \frac{(6m+5)^2}{12} \right\rfloor = \left\lfloor \frac{36m^2+60m+25}{12} \right\rfloor = \left\lfloor 3m^2+5m+2+\frac{1}{12} \right\rfloor = 3m^2+5m+2$$
$$D_3^n(2) = \left\lfloor \frac{n^2}{12} \right\rfloor$$

Therefore, in general, $D_3^n(2) = \lfloor n^2/12 \rfloor$.

How many ways are there to put exactly three 3's in a dreiben of length n, i.e. what is $D_3^n(3)$? We must choose three digit places 10^x, 10^y, and 10^z where $x \neq y \neq z$ and $0 \leq x, y, z \leq n - 1$. Now the question is, what are the possible ways to make $10^x + 10^y + 10^z \equiv 0 \pmod 7$?

In fact, there are eight cases: without loss of generality, the possible solutions are

(a) $10^x \equiv 1 \pmod 7$, $10^y \equiv 1 \pmod 7$, $10^z \equiv 5 \pmod 7$

(b) $10^x \equiv 1 \pmod 7$, $10^y \equiv 2 \pmod 7$, $10^z \equiv 4 \pmod 7$

(c) $10^x \equiv 1 \pmod 7$, $10^y \equiv 3 \pmod 7$, $10^z \equiv 3 \pmod 7$

(d) $10^x \equiv 2 \pmod 7$, $10^y \equiv 2 \pmod 7$, $10^z \equiv 3 \pmod 7$

(e) $10^x \equiv 2 \pmod 7$, $10^y \equiv 6 \pmod 7$, $10^z \equiv 6 \pmod 7$

(f) $10^x \equiv 3 \pmod 7$, $10^y \equiv 5 \pmod 7$, $10^z \equiv 5 \pmod 7$

(g) $10^x \equiv 4 \pmod 7$, $10^y \equiv 4 \pmod 7$, $10^z \equiv 6 \pmod 7$

(h) $10^x \equiv 4 \pmod 7$, $10^y \equiv 5 \pmod 7$, $10^z \equiv 5 \pmod 7$

These cases can be split into two categories. Cases (b) and (f) entail choosing one digit place from each of three sets, whereas the remaining cases entail choosing two digit places from one set and one digit place from another. To describe these two categories, let us examine cases (b) and (c).

In case (b), we are choosing one digit place each from the sets α, β, and ζ. Hence, the number of possible combinations of three digit places chosen thusly is $\binom{|\alpha|}{1}\binom{|\beta|}{1}\binom{|\zeta|}{1} = |\alpha| \cdot |\beta| \cdot |\zeta|$.

In case (c), we are choosing one digit place from α and two digit places from γ. Hence, the number of possible combinations of three digit places chosen thusly is $\binom{|\alpha|}{1}\binom{|\gamma|}{2} = |\alpha| \cdot (|\gamma|)(|\gamma|-1)/2$. Describing the other cases similarly, we find that

$$D_3^n(3) = \binom{|\alpha|}{2}\binom{|\epsilon|}{1} + \binom{|\alpha|}{1}\binom{|\beta|}{1}\binom{|\delta|}{1} + \binom{|\alpha|}{1}\binom{|\gamma|}{2} + \binom{|\beta|}{2}\binom{|\gamma|}{1}$$
$$+ \binom{|\beta|}{1}\binom{|\zeta|}{2} + \binom{|\gamma|}{1}\binom{|\epsilon|}{1}\binom{\zeta}{1} + \binom{|\delta|}{2}\binom{|\zeta|}{1} + \binom{|\delta|}{1}\binom{|\epsilon|}{2}$$

As before, since the cardinalities of the sets α through ζ are dependent on $n \pmod 6$, we would be best served writing out each of the six possible cases to see if we notice a pattern:

- If $n \equiv 0 \pmod 6$, then

$$D_3^n(3) = \frac{\left(\frac{n-6}{6}\right)\left(\frac{n}{6}\right)}{2}\left(\frac{n}{6}\right) \cdot 6 + 2\left(\frac{n}{6}\right)^3 = \left(\frac{n}{6}\right)^2\left(\frac{n-6}{2}\right) + 2\left(\frac{n}{6}\right)^3$$
$$= \frac{n^3-6n^2}{72} + \frac{n^3}{108} = \frac{5n^3-18n^2}{216}$$

- If $n \equiv 1 \pmod 6$, then

$$D_3^n(3) = \frac{\left(\frac{n-1}{6}\right)\left(\frac{n+5}{6}\right)}{2}\left(\frac{n-1}{6}\right) + \left(\frac{n+5}{6}\right)\left(\frac{n-1}{6}\right)^2 + \left(\frac{n+5}{6}\right)\frac{\left(\frac{n-7}{6}\right)\left(\frac{n-1}{6}\right)}{2}$$
$$+ 4\frac{\left(\frac{n-7}{6}\right)\left(\frac{n-1}{6}\right)}{2}\left(\frac{n-1}{6}\right) + \left(\frac{n-1}{6}\right)^3$$
$$= \left(\frac{n-1}{6}\right)^2\left(\frac{n+5}{12}\right) + \left(\frac{n+5}{6}\right)\left(\frac{n-1}{6}\right)^2 + \left(\frac{n+5}{12}\right)\left(\frac{n-7}{6}\right)\left(\frac{n-1}{6}\right)$$
$$+ \left(\frac{n-7}{3}\right)\left(\frac{n-1}{6}\right)^2 + \left(\frac{n-1}{6}\right)^3$$
$$= \frac{1}{216}(n-1)\left(5n^2-13n-10\right) = \frac{5n^3-13n^2-10n-5n^2+13n+10}{216}$$
$$= \frac{5n^3-18n^2+3n+10}{216}$$

- If $n \equiv 2 \pmod 6$, then

$$D_3^n(3) = \frac{\left(\frac{n-2}{6}\right)\left(\frac{n+4}{6}\right)}{2}\left(\frac{n-3}{6}\right) + 2\left(\frac{n+4}{6}\right)\left(\frac{n-2}{6}\right)^2 + \left(\frac{n+4}{6}\right)\frac{\left(\frac{n-2}{6}\right)\left(\frac{n+4}{6}\right)}{2}$$
$$+ \frac{\left(\frac{n-8}{6}\right)\left(\frac{n-2}{6}\right)}{2}\left(\frac{n+4}{6}\right) + 3\left(\frac{n-2}{6}\right)\frac{\left(\frac{n-8}{6}\right)\left(\frac{n-2}{6}\right)}{2}$$
$$= \left(\frac{n-2}{6}\right)^2\left(\frac{n+4}{12}\right) + \left(\frac{n+4}{3}\right)\left(\frac{n-2}{6}\right)^2 + \left(\frac{n+4}{6}\right)^2\left(\frac{n-2}{12}\right)$$
$$\left(\frac{n-8}{12}\right)\left(\frac{n-2}{6}\right)\left(\frac{n+4}{6}\right) + \left(\frac{n-2}{6}\right)^2\left(\frac{n-8}{4}\right)$$
$$= \frac{1}{216}(n-2)^2(5n+2) = \frac{5n^3-20n^2+20n+2n^2-8n+8}{216}$$
$$= \frac{5n^3-18n^2+12n+8}{216}$$

- If $n \equiv 3 \pmod 6$, then

$$\begin{aligned}
D_3^n(3) &= \frac{\left(\frac{n-3}{6}\right)\left(\frac{n+3}{6}\right)}{2}\left(\frac{n-2}{6}\right) + \left(\frac{n+3}{6}\right)^2\left(\frac{n-3}{6}\right) + 2\left(\frac{n+3}{6}\right)\frac{\left(\frac{n-3}{6}\right)\left(\frac{n+3}{6}\right)}{2} \\
&\quad + \left(\frac{n+3}{6}\right)\frac{\left(\frac{n-9}{6}\right)\left(\frac{n-3}{6}\right)}{2} + \left(\frac{n+3}{6}\right)\left(\frac{n-3}{6}\right)^2 + 2\frac{\left(\frac{n-9}{6}\right)\left(\frac{n-3}{6}\right)}{2}\left(\frac{n-3}{6}\right) \\
&= \left(\frac{n-3}{6}\right)^2\left(\frac{n+3}{12}\right) + \left(\frac{n+3}{6}\right)^2\left(\frac{n-3}{6}\right) + \left(\frac{n+3}{6}\right)^2\left(\frac{n-3}{6}\right) \\
&\quad + \left(\frac{n+3}{6}\right)\left(\frac{n-3}{6}\right)\left(\frac{n-9}{12}\right) + \left(\frac{n+3}{6}\right)\left(\frac{n-3}{6}\right)^2 + \left(\frac{n-9}{6}\right)\left(\frac{n-3}{6}\right)^2 \\
&= \frac{1}{216}(n-3)\left(5n^2-3n+18\right) = \frac{5n^3-3n^2+18n-15n^2+9n-54}{216} \\
&= \frac{5n^3-18n^2+27n-54}{216}
\end{aligned}$$

- If $n \equiv 4 \pmod 6$, then

$$\begin{aligned}
D_3^n(3) &= \frac{\left(\frac{n-4}{6}\right)\left(\frac{n+2}{6}\right)}{2}\left(\frac{n-4}{6}\right) + 2\left(\frac{n+2}{6}\right)^2\left(\frac{n-4}{6}\right) + 3\left(\frac{n+2}{6}\right)\frac{\left(\frac{n-4}{6}\right)\left(\frac{n+2}{6}\right)}{2} \\
&\quad + \frac{\left(\frac{n-10}{6}\right)\left(\frac{n-4}{6}\right)}{2}\left(\frac{n+2}{6}\right) + \left(\frac{n-4}{6}\right)\frac{\left(\frac{n-10}{6}\right)\left(\frac{n-4}{6}\right)}{2} \\
&= \left(\frac{n-4}{6}\right)^2\left(\frac{n+2}{12}\right) + \left(\frac{n+2}{6}\right)^2\left(\frac{n-4}{3}\right) + \left(\frac{n+2}{6}\right)^2\left(\frac{n-4}{4}\right) \\
&\quad + \left(\frac{n-10}{12}\right)\left(\frac{n-4}{6}\right)\left(\frac{n+2}{6}\right) + \left(\frac{n-4}{6}\right)^2\left(\frac{n-10}{12}\right) \\
&= \frac{5n^3-18n^2+12n-80}{216}
\end{aligned}$$

- If $n \equiv 5 \pmod 6$, then

$$\begin{aligned}
D_3^n(3) &= \frac{\left(\frac{n-5}{6}\right)\left(\frac{n+1}{6}\right)}{2}\left(\frac{n-5}{6}\right) + \left(\frac{n+1}{6}\right)^3 + 4\left(\frac{n+1}{6}\right)\frac{\left(\frac{n-5}{6}\right)\left(\frac{n+1}{6}\right)}{2} \\
&\quad + \left(\frac{n+1}{6}\right)^2\left(\frac{n-5}{6}\right) + \left(\frac{n+1}{6}\right)\frac{\left(\frac{n-11}{6}\right)\left(\frac{n-5}{6}\right)}{2} \\
&= \left(\frac{n-5}{6}\right)^2\left(\frac{n+1}{12}\right) + \left(\frac{n+1}{6}\right)^3 + \left(\frac{n+1}{6}\right)^2\left(\frac{n-5}{3}\right) \\
&\quad + \left(\frac{n+1}{6}\right)^2\left(\frac{n-5}{6}\right) + \left(\frac{n+1}{12}\right)\left(\frac{n-11}{6}\right)\left(\frac{n-5}{6}\right) \\
&= \frac{5n^3-18n^2+3n+26}{216}
\end{aligned}$$

Table 3 summarizes what we have learned so far about $D_3^n(2)$ and $D_3^n(3)$:

Table 3: $D_3^n(2)$ and $D_3^n(3)$

$n \pmod 6$	$D_3^n(2)$	$D_3^n(3)$
0	$\frac{n^2}{12}$	$\frac{5n^3-18n^2}{216}$
1	$\frac{n^2-1}{12}$	$\frac{5n^3-18n^2+3n+10}{216}$
2	$\frac{n^2-4}{12}$	$\frac{5n^3-18n^2+12n+8}{216}$
3	$\frac{n^2-9}{12}$	$\frac{5n^3-18n^2+27n-54}{216}$
4	$\frac{n^2-4}{12}$	$\frac{5n^3-18n^2+12n-80}{216}$
5	$\frac{n^2-1}{12}$	$\frac{5n^3-18n^2+3n+26}{216}$

Given that $D_3^n(2) = \left\lfloor \frac{n^2}{12} \right\rfloor$, it seems reasonable to suspect that $D_3^n(3) = \left\lceil \frac{5n^3-18n^2}{216} \right\rceil$. Let us see if examining some values of $D_3^n(3)$ and $\left\lceil \frac{5n^3-18n^2}{216} \right\rceil$ will enlighten us.

Table 4: $D_3^n(3)$ and $\left\lceil \frac{5n^3-18n^2}{216} \right\rceil$

n	$D_3^n(3)$	$\left\lceil \frac{5n^3-18n^2}{216} \right\rceil$
0	0	0
1	0	0
2	0	0
3	0	0
4	0	1
5	1	1
6	2	2
7	4	4
8	7	7
9	11	11
10	15	15
11	21	21
12	28	28
13	37	37
14	48	48
15	61	60
16	74	74
17	90	90
18	108	108
19	129	129
20	153	152
21	180	178
22	207	207

As evident from Table 4 above, $D_3^n(3) \neq \left\lceil \frac{5n^3-18n^2}{216} \right\rceil$, but they seem very similar. Determining the relationship between $D_3^n(3)$ and $\left\lceil \frac{5n^3-18n^2}{216} \right\rceil$ may be a topic for future research, but is beyond the scope of this paper.

Finding $D_3^n(i)$ for greater values of i becomes increasingly difficult, perhaps even impractical. It is beyond the scope of this paper, but should a general form for $D_3^n(i)$ be found, one could, from a combinatorial perspective, corroborate the results of the second section, allowing one to find a different form for the explicit equation of $A_7^i(n)$. In other words, one would have yet another method to determine the number of dreibens of length n divisible by 7.

CONCLUSION

While we are still far from determining whether the set of prime dreibens is finite or infinite, our exploration of dreibens modulo 7 has cast much light on the problem. Now, we have several methods to determine how many dreibens of length n are divisible by 7, therefore allowing these dreibens to be ruled out during our search for prime dreibens. We have also opened up several opportunities for inquiry. For instance, explicit matrices for A^n where $n \equiv 1, 2, 3, 4, 5 \pmod{6}$ must still be determined. The relationship between $D^n_3(3)$ and $\left\lceil \frac{5n^3-18n^2}{216} \right\rceil$ remains a promising mystery, and we have yet to find a generalized formula for $D^n_3(i)$. Although dreibens remain a mysterious type of number, current research in dreibens show promise in both uncovering new mathematical problems and achieving our ultimate goal of finding infinitely many prime dreibens, if they exist.